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Asymmetrical dripping

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Dripping from a faucet is studied, where by cutting diagonally the tip breaks the cylindrical symmetry. We measured long dripping time series recorded with a laser-beam apparatus and continuous change in drop position using a high-speed camera. It is found that the added degree of freedom produces a transversal oscillation of a pending drop, which couples with a vertical oscillation induced by the breaking off of the previous drop. As a consequence dripping times shorten and dripping patterns regularize. The effect is attributed to the "reduced" contact circle and to decreased frequency of the vertical oscillations of the residue. A very complex flow circulation pattern within the forming drop is observed. It is suggested that geometrical shape vibrations of the pending drop take place by the development of eddies of different amount, more effective in the vicinity of the highest edge of the tip, where they dissipate more slowly. This asymmetrical liquid flow is brought about by the slanting form of the orifice and overlaps with the axial oscillations.

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I. INTRODUCTION

Dripping from a faucet is a phenomenon investigated both experimentally [1–23] and theoretically [1,23–37]. These studies have established the dripping faucet as a sort of paradigmatic system for chaos. On the other hand, the formation of liquid drops is of interest in a wide variety of engineering applications, such as distillation and extraction processes, ink-jet printing, and biochip arrayers [35].

Up to now, experiments are generally performed by allowing a liquid to drip from a vertical tube of cylindrical form, so that the dripping dynamics is axially symmetric. The asymmetries that eventually are experimentally observed can be attributed to the delicate dependence of dynamics upon small deviations of the nozzle from the cylindrical form, so that they represent the symptom of a strong dependence of the system from nonaxial perturbations. Recently, a confirmation of such a conjecture was given in Ref. [21], where an investigation of the dripping from an inclined nozzle was performed. The inclination angle of the nozzle can constitute an effective control parameter. However, such an experiment introduces too many alterations with respect to the dripping from a vertical nozzle if the inclination angle is large, and the comparisons with the dripping from an identical vertical tube become tenuous. On the other hand, we know that very small changes of the form of the nozzle tip are sufficient to perturb the system, changing its dynamical behavior. Recently, we investigated the axially symmetric perturbations on the dripping behavior introduced by using cylindrical nozzles with tips of different geometric forms [20]. Such studies confirm the strong dependence of the dynamics upon even weak axially symmetric variations. In fact, nozzles with tips of different axisymmetrical geometry affect the wetting characteristic of hanging liquid because they determine the three-phase contact line. Measurements of dripping from nozzles with flat and bevelled tip of different diameters show that the formation of liquid drops at low flow rate depends upon the nozzle size, liquid properties, wall thickness, and flow rate [8,20].

The introduction of an asymmetrical perturbation, obtained by cutting obliquely the tip of the nozzle, breaks the cylindrical symmetry and produces changes in the topological properties of the experimental attractors. The analysis of such asymmetrical dripping adds to the understanding of the dynamics of drop formation, which is of great importance in technological applications [8,35] and relevant to the study of biological systems [21].

In this work, we performed experimental investigations along two complementary ways: dripping time measurements by laser-beam apparatus and analysis of drop formation by means of high-speed video recording. We aimed at probing the effects of geometrical parameters on the features of drop formation without disregarding their influences on long-time dripping behavior. In order to do this we compared results on measurements of the dripping of water from nozzles with flat (F) and obliquely (O) shaped cut tips.

The paper develops as follows. In Sec. II the experimental procedure is described. Preliminary measurements, presented in Sec. III, were performed in order to guarantee almost exactly the same flow rate for the various nozzles we used. Dripping bifurcation diagrams from F and O nozzles are compared in Sec. IV. Vertical oscillations of pending drops from axisymmetric flat tip nozzles are analyzed in Sec. V A, whereas in Sec. V B the analysis of the drop formation at three different cut angles of the tip is presented. Conclusions are drawn in Sec. VI.

II. EXPERIMENTAL PROCEDURE

The experimental apparatus is shown in Fig. 1(a). Dripping times were recorded when drops interrupted the beam of a laser-beam apparatus. We measured the time interval t elapsed between the instant two successive drops reach the

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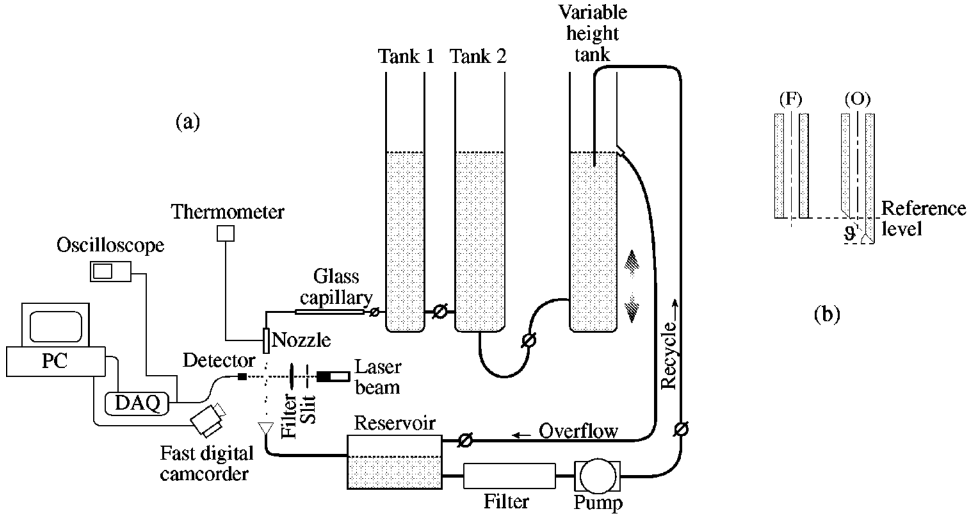

FIG. 1. (a) Schematic diagram of the experimental setup. (b) Configuration of the stainless nozzles utilized: Flat (*F*) and oblique (*O*) shaped.

laser beam, which represents the formation time of the second drop. With our apparatus dripping times as short as 50 ns can be detected. Constant flow rates are obtained by means of overflows. In fact, the use of needle valves makes the time needed to reach a steady flow longer [3]. Setting the overflow at different heights of the water column varies the flow rate, which depends solely on the height of the liquid level in the tank 1 [Fig. 1(a)]. By closing the stopcock between tanks 1 and 2 the tank 1 continuously empties, and bifurcation diagrams are obtained, where dripping intervals t are plotted as a function of the flow rate Q.

Drop formations at low flow rate were also recorded with a digital video camera, or, if necessary, with the aid of an associated high-speed motion analysis/video system, the Kodak Motion Corder Analyzer Model "Photron Fast Cam Super 10 K." The position of the center of mass of the drop oscillating under the nozzle was estimated from the shape of the digitized drop images.

Measurements are performed at the constant temperature of 22.0 °C within an interval of 0.1 °C. We used steel nozzles of inner diameters $d=2.0, 4.0$ mm (wall thickness of 1.0 mm) with flat and cut tips (cut angle $\vartheta=15°, 30°, 45°$ with respect to the horizontal plane).

III. PRELIMINARY MEASUREMENTS

In order to compare dripping properties of O and F tip nozzles, the bottoms of the two faucets were set at heights so that their flow rates Q were almost the same when the overflow is fixed. In fact, in principle, at a fixed level of the overflow, the flow rates Q of the two nozzles can slightly depend upon the minimal difference (of the order of about 2 mm) at the exit due to the cut [see Fig. 1(b)].

Preliminary measurements were carried out in order to estimate the influence upon the dripping rates of minimal difference on the reference level of the tube exit height of a F tip. Measurements were performed by choosing overflow levels in an interval corresponding to flow rates values ranging from ~0.15 ml/s to ~0.28 ml/s. At each fixed overflow level we compared the Q values by slightly changing the reference level of the F nozzle. We found flow rate differences, per millimeter of change of the reference level, ranging between ~0.7% and ~0.3%. Thus, a lower level of the tip somewhat increases Q. On account of this, we placed the nozzles as shown in Fig. 1(b), so as to avoid a reduction of the flow rate of the O nozzle due to the level difference.

Later, flow rates versus the overflow level were also compared for F and O nozzles set in the position of Fig. 1(b), where one can see that the O nozzle has the exit tip at a level slightly lower than the F nozzle. Nevertheless, we observed that, at a fixed value of the overflow height, generally the measured flow rate of the oblique cut orifice was slightly lower than the similar flat one, within the experimental uncertainties. Exceptions can eventually happen at low Q values where the minimal difference at the exit become relevant or in the ranges of Q where dripping is irregular or chaotic.

IV. BIFURCATION DIAGRAMS

In order to investigate the characteristics of dripping from the nozzles utilized, we performed dripping times measurements as a function of Q (*dripping spectra*).

We also counted the drop numbers $n(F)$ and $n(O)$, respectively, from F and O tips, when the tube 1 is left to empty from the same level (by disconnecting the tube from the other ones, joined in parallel). We found in such circumstances $n(O)>n(F)$; so the mean drop mass $m(O)<m(F)$ and O dripping times are, on average, lower than the corresponding F ones. The total measured drops number, corresponding to a fixed depleted mass of water, give $n(O)$ greater than $n(F)$ of about 10%; the corresponding emptying times T are in the relation $T(O)\geqslant T(F)$, with difference of only ~0.3%. Furthermore, the video-camera observations show that, at the same Q, the O nozzles have an "effective" contact circle with a radius smaller than the F one (see Fig. 2). As a consequence, the time scale for the formation of a pendent

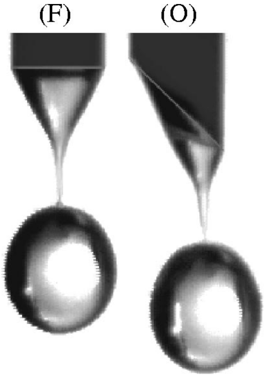

FIG. 2. Photographs showing at $Q \sim 0.1$ ml/s the breakup of a drop falling from F and O nozzles. We can see that the "effective" contact circle for the O tip is smaller than the F one.

drop ($\propto d^3/Q$) and the pinch-off times [$\propto(\rho d^3/\sigma)^{1/2}$, where ρ and σ are, respectively, the density and the surface tension] [22] of the O nozzle can be different from the F ones. The measured emptying times T and drops numbers n give (for O with $\vartheta=45°$) a mean time scale ratio $[T(O)/n(O)]/[T(F)/n(F)] \cong 0.9$ for the formation of a drop (for the other cuts this value is closer to 1). However, we found that these small variations cannot be enough to explain the different behavior observed in our dripping spectra, as the differences of the time scale should be obtained by substituting the O nozzle with a F nozzle of appropriate diameter, which, nevertheless, does not give similar plots for the bifurcation diagram. Therefore the different behaviors of the O nozzles cannot be attributed only to the reduced contact circle.

Previous observations refer to emptying times, and thus to average flow rates. At arbitrary fixed values of Q these considerations cannot be valid, in particular, when the dripping is irregular. Thus the perturbation introduced by the cut does not reduce the relevance of Q, which remains everywhere the principal control parameter of the system.

Bifurcation diagrams of dripping from F and O nozzles ($\vartheta=15°, 30°, 45°$) with $d=2$ mm are shown in Fig. 3. The O dripping diagrams evolve through an alternate period-1 motion (which occasionally shows rings of doubling period) and reveal a more regular behavior over the whole flow rate range, with respect to the F diagram, with the exception for the high flow rate values (not reported in Fig. 3; see, however, Figs. 5 and 7). In fact, the F spectrum shows a roughly periodic behavior, but the period-1 states are masked by the alternating belts of chaotic structures [Fig. 3(a)]. Instead, the dripping times decrease (in agreement with the previous emptying times analysis) and the number of the repeating structures reduces manifestly for the O spectra with different characteristics of behavior that become more evident passing from $\vartheta=15°$ to $\vartheta=45°$.

The pattern evolution as a function of ϑ, illustrated in Fig. 3, shows significant changes passing from $\vartheta=0°$ to $\vartheta=15°$, the repeating series of chaotic states evolves toward an analog structure, but with lower number of chaotic sequences and a reduced amount of irregularity. Increasing further ϑ, patterns evolve reducing the width of irregular states, but

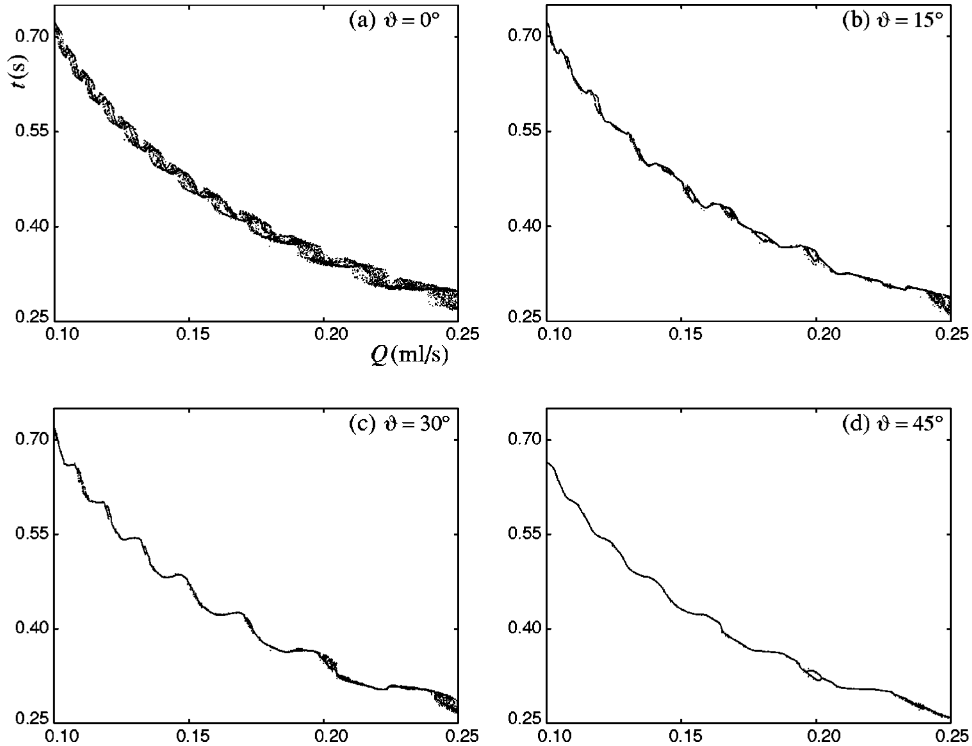

FIG. 3. Plots of dripping bifurcation diagrams for nozzles with orifice of diameter $d=2$ mm: (a) F nozzle ($\vartheta=0°$); and O nozzles with (b) $\vartheta=15°$, (c) $\vartheta=30°$, (d) $\vartheta=45°$.

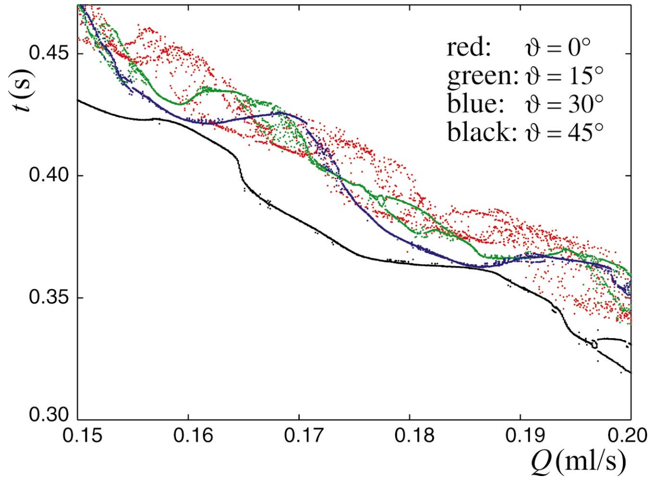

FIG. 4. (Color) Blow-up of the superimposed bifurcation diagrams shown in Fig. 3.

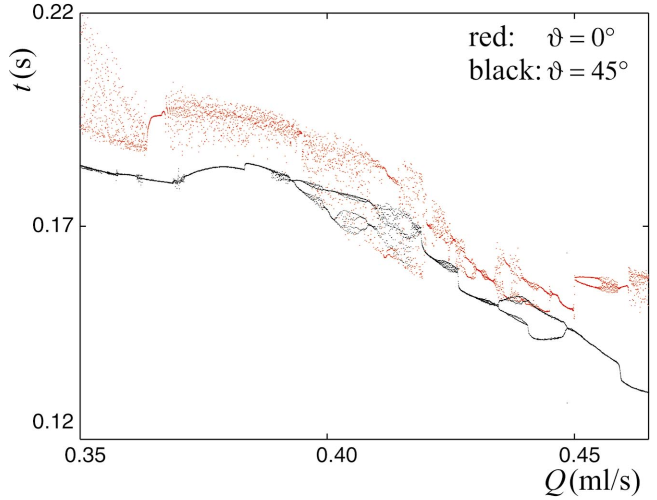

FIG. 6. (Color) Enlargement of the superimposed bifurcation diagrams shown in Fig. 5.

maintaining unchanged the gross structure of oscillating behavior. At $\vartheta = 45°$ a period-1 state is observed on almost the whole range of Q shown. In Figs. 3(c) and 3(d) we can also observe large regions where, by increasing Q, the patterns of period-1 states change from an anomalous rise [16] to a regular decrease, whereas for a F tip this behavior happens only at low values of Q and/or for appropriate orifice diameters [20].

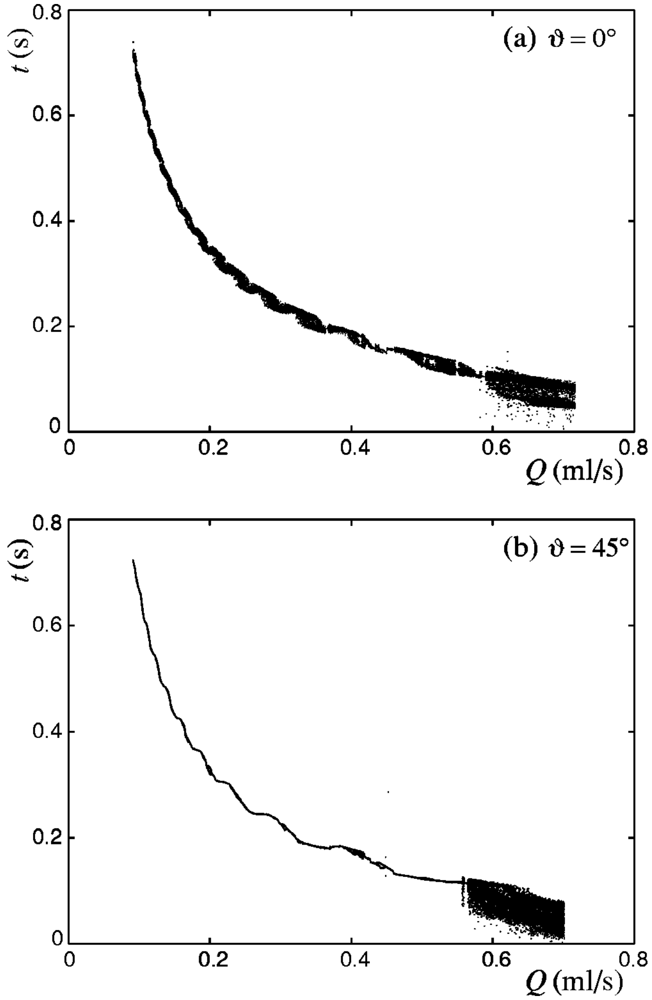

FIG. 5. Plots of dripping spectra for nozzles of $d=2$ mm: (a) $\vartheta=0°$; (b) $\vartheta=45°$.

Inspection of the previous diagrams reveals that even if the mean dripping time decreases as the angle cut ϑ is increased, however, there are many regions where, at a fixed Q, dripping times of O nozzles are greater than the corresponding F ones. This is highlighted in Fig. 4, where a blow-up of the superimposed spectra of Fig. 3 is shown for a particular flow rate region.

The great diversity of the bifurcation diagrams between F and O dripping is evident by looking at Fig. 5, where the diagrams are plotted on a wide range of Q. It is interesting also to observe the enlargement of Fig. 6, where one can see flow rate regions where O dripping spectrum retains the memory of the coarse behavior of F dripping and a region, around $Q=0.45$ ml/s, where F pattern is broken undergoing a crisis.

When the diameter of the tip is greater than $d=2$ mm, also F nozzles show dripping spectra with numerous zones of repeating behavior [20]. In Fig. 7 the analog of the spectra of Fig. 5, for an orifice with $d=4$ mm are reported. As previously noted, the period-1 states are mostly included between zones of chaotic states. What has to be marked is that, in this case, the pattern changes of the O spectrum with respect to the F ones are less manifest, because of the less important influence of the cut on the dripping when the diameter of the orifice increases.

By denoting with Q_n the flow rates at the nth transition point at which the period-1 state undergoes (with increasing Q) a transformation or it takes a maximum, and with t_n the corresponding dripping time intervals, we found that t_n decrease as $t_n \propto Q_n^{-\alpha}$. In all cases, linear fits on a log-log scale of time intervals t_n against Q_n give mean slopes $\alpha \lesssim 1$. There results a mean growth of the drop mass with Q [16], even if a decrease of the mass in limited intervals of Q can occur.

We can summarize our considerations about the observed bifurcation diagrams by stating that, as the cut angle is in-

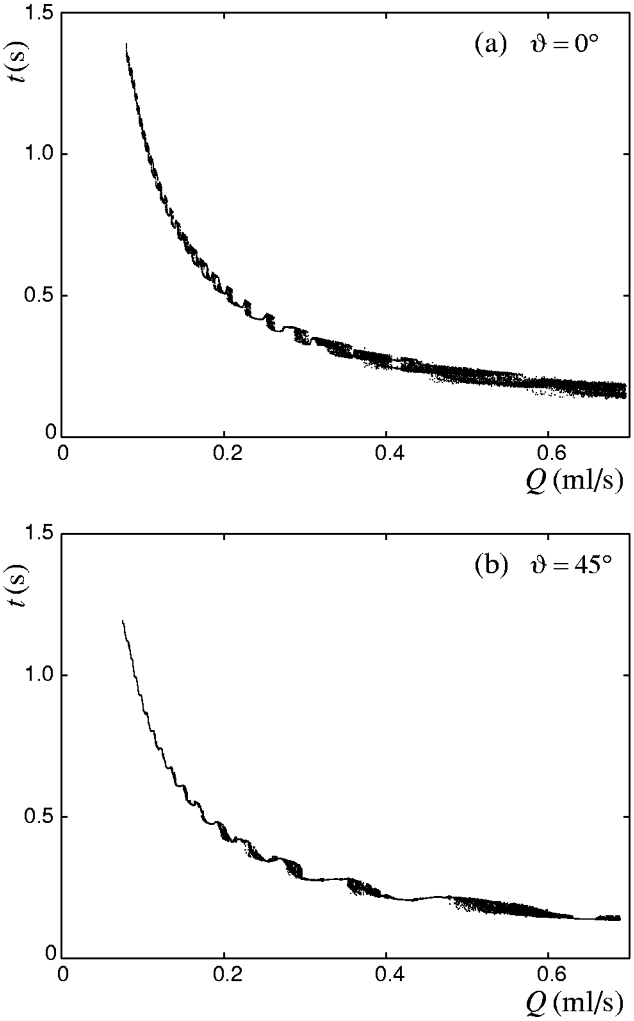

FIG. 7. Same as in Fig. 5, but for $d = 4$ mm.

creased, one finds the following: (i) more regular dripping spectra, (ii) smaller dripping times and drop masses, (iii) lower frequency of the alternate periodic-irregular structures.

V. DROP OSCILLATIONS

As the dynamic evolutions, shown by the bifurcation diagrams of O and F tips, appear rather different, we observed drop formation also with a fast camcorder.

When a drop hangs from a faucet it grows due to the continuous flow of water and it oscillates subjected to the weight and to the surface tension, which acts as a restoring force. In this process a thin liquid bridge (between the pendent mass and the nascent drop) forms till a drop detaches and falls, stimulating a rebound and mechanical vibrations in the residual water, affecting the time of formation of the next drop. The falling drop also oscillates, retaining memory of its own previous hanging behavior. Therefore it is of great interest to analyze the oscillation of the position of the center of mass of the drop under the nozzle.

A. Symmetrical oscillations

During the drop formation process, the effects of the overhanging pushing liquid and viscosity, and the increased weight of the pending drop slow the oscillator motion, leading to the formation of the neck. After the drop leaves the faucet, at the pinch-off, it creates axial symmetric oscillations in the residue. This behavior is shown in Fig. 8, where ten different photograms of a drop pending from a F tip of 2 mm of diameter are reported. From the images we can see that, after the drop falls, vibrations of the residue liquid with smaller amplitude overlap the oscillatory motion, and appear as a swelling in proximity of the edge of the flat surface. In fact, the complex flow patterns that originate inside the forming drop can give rise to eddies or zones of fluid recirculation [17]. The liquid elements which feed the drop impinge on the drop surface at the bottom, then flow tangentially up and produce an annular protuberance on the upper part of the drop. In an ideally axisymmetrical tube flow patterns inside the drop retain anyway cylindrical symmetry.

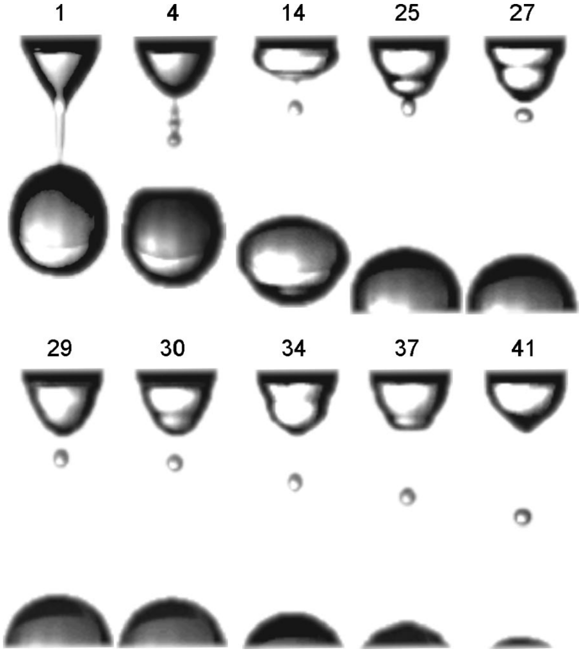

FIG. 8. A sequence of pictures showing the F axial symmetric oscillations of the liquid residue and a falling drop at $Q \sim 0.1$ ml/s. One observes also a satellite drop. On each image we report the frame number captured by the high-speed video system, operating at 2000 fps.

In Fig. 9 the position of the center of mass of a forming drop, reconstructed by the digitized images, is reported for three different values of Q. As Q increases dripping time decreases, according to spectra patterns, break point position also increases, as gravity force quickly takes over; owing to the augmented driving force of the pressure the widths of oscillations are larger and frequency lower. The amplitude and the frequency of the oscillatory motion of each forming drop decrease until the neck forms.

B. Asymmetrical oscillations

When a drop hangs from a cylindrical tube with the tip cut obliquely, the further freedom degree breaks the cylindrical symmetry so that oscillations in the transverse direction become possible.

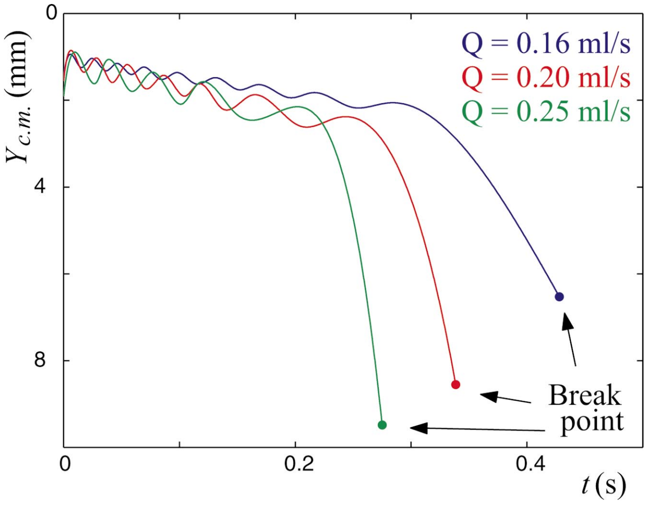

FIG. 9. (Color) Vertical position of the center of mass vs time of a forming drop from a *F* nozzle of diameter $d=2$ mm, for three values of flow rate. The origin of the coordinate system is located at the intersection point of the vertical axis of symmetry with the horizontal plane. The curves were obtained by spline interpolation among the data points.

Ten different photograms of a drop pending from an *O* tip (with $\vartheta=45°$) are shown in Fig. 10. The images sequence shows clearly the presence of oscillations in the transverse direction. From the analysis of the video recording, we deduced the following results. After a drop falls, the residue is subjected to the action of a further competing transversal oscillation. This motion can be attributed to the increased action of the surface tension in the transversal direction, which pushes the drop to assume a semispherical form with the basis set at the oblique surface of the tip. Initially, the competing action among nonsymmetrical surface tension, pressure, and weight force, produces a sort of waving motion, which starts from the bottom of the drop and overlaps to the liquid entering the drop from the tube (frames 11 and 17 of Fig. 10). These asymmetrical flow fields are the analog of those axisymmetrical ones observed for the *F* nozzle (Fig. 8).

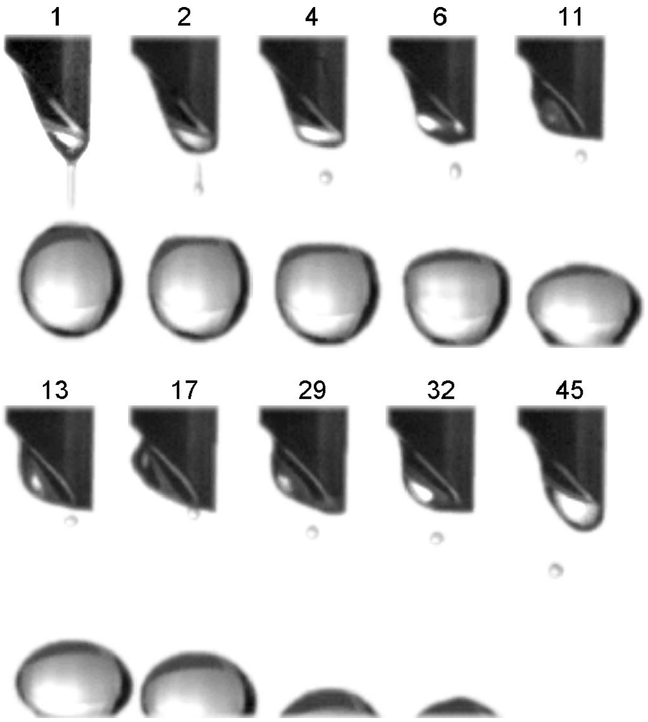

FIG. 10. The analog of Fig. 8 for the oscillations of a forming drop from an *O* nozzle with $\vartheta=45°$.

For a *F* tip the recirculation of fluid within the forming drop redirects symmetrically upward along the surface, encompassing the entire wall of the drop. Instead, for the *O* tip there is a different behavior of the recirculating liquid elements, which go upward with higher velocity toward the upper border of the cut, where the transversal component of surface tension works and the quantity of fluid entering the drop from the tube is lower. As a consequence, the motion of the center of mass of the forming drop can be described as a combination of two oscillations in the plane perpendicular to the cut surface and containing upper and lower cut points.[1] We observed that, as the mass of the drop increases, the transversal motion decreases, and only longitudinal motion survives before the neck forms. Furthermore, we observed secondary drops which, after colliding with the liquid hanging from the tube, are thrown sideways along a direction at right angles with respect to the elliptical free surface of the opening (and of the forming drop).

Owing to the complicated motion of the various elements of the fluid inside the drop, the description of the irregular motion of the drop in terms of center of mass coordinates simplifies and clarifies the analysis of the phenomenon. In Fig. 11 the coordinates of the drop center of mass reconstructed from the digitized images[2] are drawn at two different flow rate values and three values of ϑ. The comparison at the same flow rate between the longitudinal periods of *O* and *F* nozzles shows generally a lower frequency for the oblique nozzles. The effect is more evident for the higher ϑ [Figs. 11(c)–11(f)], where the weight of the pending drop overcomes more easily the vertical component of restoring surface tension force, thus confirming the observation of lower dripping times. For an *O* nozzle, as *Q* increases, the amplitude of vertical oscillations slightly enhances and their frequencies decrease, such as the *F* nozzle. The transversal oscillations are in phase opposition with the corresponding vertical ones, at least for $\vartheta=30°$ and 45°.

Thus the dynamics of drop formation suggests that added degree of freedom changes the total potential energy (i.e., gravitational plus surface) [23], so that the drop reaches the breaking point with reduced vertical frequency resulting in a weakening of the correlation with the next drop and making the gravity force more important with respect to the restoring

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[1] Observe that dripping is symmetric for reflection with respect to this plane. So the drop center of mass ideally moves on such a plane.

[2] The irregular movements of the fluid elements within the forming drop induce uncertainties in the evaluation of the drop position. They influence our estimates of the amplitude of drop oscillations but not appreciably their frequencies.

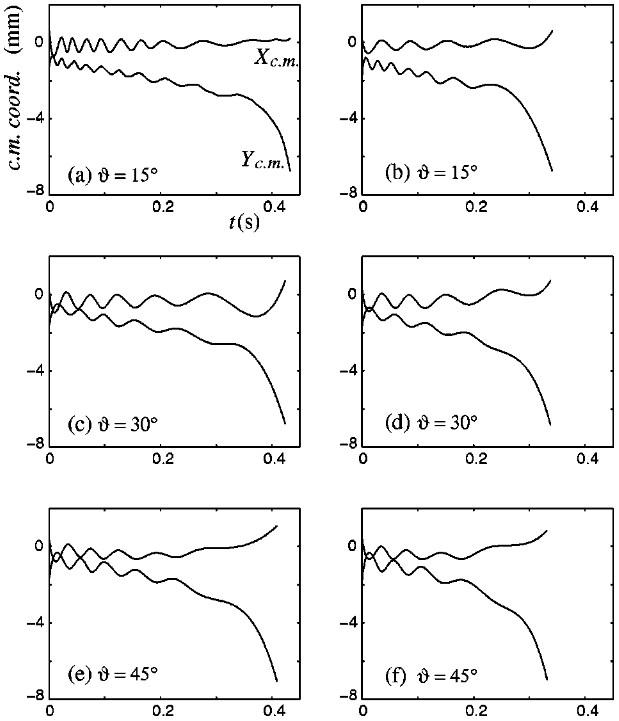

FIG. 11. Center of mass coordinates (origin at the intersection point of the vertical axis of symmetry with the cut plane) vs time of a drop hanging from an O nozzle ($d=2$ mm), at two flow rates: $Q=0.16$ ml/s (a), (c), (e); $Q=0.20$ ml/s (b), (d), (f).

force of the surface tension. The result is a smooth period-1 behavior on a plentiful range of the flow rate, as is shown in the plots (b) of Figs. 5 and 7, where it is evident that the asymmetry produces a bifurcation pattern with a reduced frequency of the alternating periodic and irregular states.

VI. CONCLUSIONS

We have investigated the dynamical effects introduced by cutting diagonally the tip of a nozzle in a dripping faucet experiment, both through long-time dripping and single-drop formation analysis. Cutting obliquely the tip of a faucet breaks the cylindrical symmetry of the nozzle and introduces correlation effects on the dripping time series and on the dynamical properties of the forming drop. The dripping time series, recorded at variable flow rate, gives bifurcation diagrams that, increasing the nozzle cut angle, show patterns of increasing regularity. The reason can be found by looking into the single-drop formation process. Depending on physical and geometrical parameters of the experimental setup, different flow patterns inside a forming drop were experimentally observed. A drop pending from a flat nozzle increases undergoing damped oscillations induced by the breaking of the previous one. Flow fields inside the forming drop create a recirculation of the liquid elements entering from the tube toward the apex; such elements flow up along the surface and form an annular wave near the flat surface of the tip. When the flow rate is low the liquid eddies disappear and the oscillations damp long before the entering fluid forms the next drop, so that dripping is regular. Increasing Q the recoil of the residue competes rapidly with the impinging flow, so that irregular dripping results. If the cylindrical symmetry is broken, with an oblique cut, two effects go on: the liquid eddies develop asymmetrically (increasing in size along the highest point of the tip) and the drop develops with a reduced effective contact circle. As a result the vertical part of oscillatory motion decreases in frequency so that the weight widely overcomes the restoring force (overdamping the oscillations). The breakup moment advances and a more regular dripping occurs.

However, without a suitable contrast medium, these liquid circulations are particularly hard to observe. Studies are in progress in order to make these investigations. They can also shed some light on the anomalous raising of the dripping times observed in the flow rate range where drops fall periodically.

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